ζ-irrationality search: after a golden section approach, another esthetic but vain attempt.

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Abstract

The proof of the irrationality of ζ(5) is a long standing open problem. The present paper abandons a golden section inspiration (as many artists may have done in their field), and suggests a different approach, based on the esthetic value of a formula. Yet, it appears as vain as the first one.

1. ζ(2), ζ(3) and the golden section.

Although a previous paper was at first sight but a summary of existing proofs for the irrationality of π, ln2, $\zeta(2) = 1+\frac{1}{2^2}+\frac{1}{3^2}...$ and $\zeta(3) = 1+\frac{1}{2^3}+\frac{1}{3^3}...$ (see [3]), it was attributed the "Lester Ford Award 2002" by the Mathematical Association of America, while some found an inspiration in it for a query about still other famous mathematical constants, such as e and Euler's constant (see [6]), and others continued their computer search for similar constants (see [5]). To F. Beukers (see [1]), the reason for these reactions was the lack of progress in this field at the time, and thus any sensible new impulse is meaningful.

Furthermore, there was a link to mathematical notions used more often in artistic circles, though not so well known to pure number specialists: the golden section, noted ϕ, τ, g or σ_{Au}, and the silver and bronze sections, σ_{Ag} and σ_{Br}. They are the positive roots of $x^2-nx-1=0$, for n=1, 2, 3…(see [7]), and they emerged as follows in the explained proofs.

For ζ(2), $0 < \left|\int_0^1\int_0^1 \frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}}dxdy\right| = \left|\frac{R_{n+1}+S_{n+1}\zeta(2)}{T_{n+1}}\right|$ for any n∈N, while $M_2=\left|\max(\frac{x(1-x)y(1-y)}{1-xy})\right|$ and $(M_2)^n.T_{n+1} \le (M_2)^n.(3^{n+1})^2 \le 1$. Thus, the rationality of ζ(2) would lead to a contradiction as R_n and S_n (and T_n) are integers and $|R_{n+1}+S_{n+1}\zeta(2)| \to 0$.

For ζ(3), $0 < \left|\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)y(1-y)z(1-z))^n}{(1-(1-xy)z)^{n+1}}dxdydz\right| = \left|\frac{R_{n+1}+S_{n+1}\zeta(3)}{T_{n+1}}\right|$ for any n∈N, while $M_3=\left|\max(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z})\right|$ and $(M_3)^n.T_{n+1} \le (M_3)^n.(3^{n+1})^3 \le 1$. Thus, the rationality of ζ(3) would lead to a contradiction as $|R_{n+1}+S_{n+1}\zeta(3)| \to 0$.

As $M_2=|(1/\sigma_{Au})^5|$ is attained for $x=y=-1/\sigma_{Au}$, and $M_3=|(1/\sigma_{Ag})^4|$ for $x=y=-1/\sigma_{Ag}$, it was expected that $\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)w(1-w))^n.(1-xy))^{n+1}}{((1-(1-xy)z)(1-(1-xy)w))^{n+1}}dxdydzdw\right|$ had potential for attempting a proof for $\zeta(4)$ (and its extension, eventually, for $\zeta(5)$), because the M_4-maximum was attained for $x=y=-1/\sigma_{Br}$. Still, the same paper also pointed out this option failed since the integral is not of the form $(R_{n+1}+S_{n+1}\zeta(4))/T_{n+1}$. Thus, the golden-silver-bronze section connection was misleading (partially, see [4] - but this happened in art too: see [2]).

2. Another approach for a ζ-irrationality proof.

An esthetic expression, based on the logic in the form of the integrand in the given proofs, seemed promising to overcome some surprising difficulties of ζ-irrationality proof attempts:

(E) $\qquad \zeta(m) = \frac{1}{m-1}\int_0^1\int_0^1\int_0^1\ldots\int_0^1 \frac{1}{(1-xy)(1-xyz)\ldots(1-xyz\ldots w)}dxdydz\ldots dw$.

Now, the proof could go by checking the following conjectures:

$$\text{(I)}\ \ 0 < \left|\int_0^1\int_0^1\int_0^1\ldots\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)\ldots w(1-w))^n}{\left((1-xy)(1-xyz)\ldots(1-xyz\ldots w)\right)^{n+1}}dxdydz\ldots dw\right| = \left|\frac{R_n+S_n\zeta(m)}{T_n}\right|,\ R_n, S_n, T_n \in \mathbb{Z}.$$

$$\text{(II)}\ \ M_m = \max\left|\frac{x(1-x)y(1-y)z(1-z)\ldots w(1-w)}{(1-xy)(1-xyz)\ldots(1-xyz\ldots w)}\right| \text{ with } \left|M_m.3^m\right|\le 1.$$

- For $\zeta(2)$, the proposal coincides with the well-known proof.

- For $\zeta(3)$, there are differences with Beuker's proof, at first sight, but the substitution $q=y\frac{1-x}{1-xy}$, that is, $y=\frac{q}{1-x+xq}$, transforms the proposed integral in a Beuker's type, as given in §1:

$$\left|\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)y(1-y)z(1-z))^n}{\left((1-xy)(1-xyz)\right)^{n+1}}dxdydz\right| = \left|\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)q(1-q)z(1-z))^n}{\left(1-(1-qz)x\right)^{n+1}}dxdqdz\right|.$$

This proves expression (E), and the fact that M_3, the maximum of $\frac{x(1-x)y(1-y)z(1-z)}{(1-xy)(1-xyz)}$, is identical to Beuker's $|(1/\sigma_{Ag})^4|$ = 17-12√2 value. It equally proves the validity of the approach for $\zeta(3)$ with the additional "improvement" that the initial integral is more elegant, but this of course is a subjective matter.

- For $\zeta(4)$, the (very large) algebraic expression for the maximum value M_4 has no more relation to the bronze mean but, numerically at least, condition (II) could be verified: $M_4.3^4<1$; that is a good start.

The substitutions $q=y\frac{1-x}{1-xy}$ and $r=z\frac{1-xy}{1-xyz}$, that is, $y=\frac{q}{1-(1-x)q}$ and $z=\frac{(1-(1-x)q)r}{1-(1-x)qr}$ lead to:

$$\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)w(1-w))^n}{\left((1-xy)(1-xyz)(1-xyzw)\right)^{n+1}}dxdydzdw\right|$$

(E2) $$= \left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)q(1-q)r(1-r)w(1-w))^n}{\left(1-(1-qrw)x\right)^{n+1}}dxdqdrdw\right|.$$

For $n=0$, it establishes the expression (E) in a similar way as in [1]. Indeed, the expression

(E3) $$\left|\int_0^1\int_0^1\int_0^1 \frac{q^{j+s}r^{k+s}w^{m+s}}{1-qrw}dqdrdw\right| = \left|\sum\frac{1}{(j+s+1)(k+s+1)(m+s+1)}\right|$$, becomes, for identical values:

$$\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{q^{j+s}r^{j+s}w^{j+s}}{1-qrw}.\log(qrw).dxdqdrdw\right| = \left|\sum\frac{-3}{(j+s+1)^4}\right|.$$

Thus, $$\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{1}{(1-xy)(1-xyz)(1-xyzw)}dxdydzdw\right| = \left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{1}{1-(1-qrw)x}dxdqdrdw\right|$$

$= \left|\int_0^1\int_0^1\int_0^1 \frac{1}{1-qrw}\log(qrw)dqdrdw\right| = \zeta(4)$, that is, the expression (E) for m=4.

An approach similar to Beuker's for the ζ(3) case, can be applied on (E2):

$$\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)q(1-q)r(1-r)w(1-w))^n}{(1-(1-qrw)x)^{n+1}}dxdqdrdw\right|$$ partial integration, n times:

$$= \left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{((1-x)(1-q)(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n}{1-(1-qrw)x}dxdqdrdw\right|$$ subs x=(1-p)/(1-(1-qrw)x) :

$$= \left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(p(1-q)(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n}{(1-(1-qrw)p)^{n+1}}dpdqdrdw\right|$$ partial integration, n times:

$$= \left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(1-q)^n \frac{1}{n!}\frac{d^n}{dr^n}(r(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n}{1-(1-qrw)p}dpdqdrdw\right|$$ a final integration:

$$= \left|\int_0^1\int_0^1\int_0^1 \frac{(1-q)^n \frac{1}{n!}\frac{d^n}{dr^n}(r(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n}{1-qrw}\log(qrw).dqdrdw\right|$$

Already for n=1, it is seen that

$$(1-q)^n \frac{1}{n!}\frac{d^n}{dr^n}(r(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n = (1-q)(1-2r)(1-2w)$$

$= 1 - q - 2r + 2qr - 2w + 2qw + 4rw - 4qrw$,

and although the terms 1 and $-4qrw$ yield $-3\zeta(4)$ and a fraction, the other terms create a non-vanishing ζ(3) expression, as the expression (E3) shows using a procedure similar to Beuker's proof:

$$\left|\int_0^1\int_0^1\int_0^1 \frac{q^j r^k w^m}{1-qrw}\log(qrw)dqdrdw\right| = \begin{cases}\text{a fraction, if all j, k, m are different}\\ \zeta(4)\text{ and a fraction, if all j, k, m are equal}\\ \zeta(3)\text{ and a fraction, if exactly two j, k, m are equal}\end{cases}$$

Thus, the above calculations only show that $0 < \left|R_{n+1} + S_{n+1}\zeta(3) + U_{n+1}\zeta(4)\right| \to 0$, for R_n, S_n, U_n ∈Z. The only thing to remember from the present paper may be the general esthetic expression (E) for ζ(m), but, alas, the author did not have the nerve to check if this expression is not well-known and merits a proof.

References

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